UTHEP-299, April 1995

The Pion-Nucleon Coupling Constant in QCD Sum Rules

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(February 1, 2008)

Abstract

The pion-nucleon coupling constant $g_{\pi N}$ is studied on the basis of the QCD sum rules. Both the Borel sum rules and the finite energy sum rules for $g_{\pi N}$ are used to examine the effects of higher dimensional operators (up to dim. 7) and α_s corrections in the operator product expansion. Agreement with the experimental number is reached only when S_π/S_N is greater than one, where S_π (S_N) is the continuum threshold for the $g_{\pi N}$ (nucleon) sum rule.

I. INTRODUCTION

The pion-nucleon coupling constant $g_{\pi N}$ is one of the most fundamental quantities in hadron physics. $g_{\pi N} = 13.4 \pm 0.1$ is obtained empirically using the $N - N$ scattering data, $\pi - N$ scattering data and the deuteron properties [1]. From the theoretical point of view, it is a great challenge to reproduce this value from the first principle namely the quantum chromodynamics (QCD). So far, there have been two attempts; one is based on the lattice QCD simulations (see ref. [2]) and the other is based on the QCD sum rules (QSR) [3,4]. The latter approach for $g_{\pi N}$, however, has not been explored in great detail beyond the leading order of the operator product expansion (OPE) [5,4] within the authors' knowledge.[1] The purpose of this paper is to reexamine the problem using the currently available information on the higher dimensional operators in OPE and the α_s corrections to the Wilson coefficients.

To study $g_{\pi N}$ in QSR, two methods have been proposed so far: (i) method based on the two point function $\langle 0 \mid \mathrm{T}\eta(x)\bar{\eta}(0) \mid \pi \rangle$, and (ii) method based on the three point function $\langle 0 \mid \mathrm{T}\eta(x)\bar{q}i\gamma_5 q(y)\bar{\eta}(0) \mid 0 \rangle$, where $\eta(x)$ is the nucleon interpolating field. We will take the first approach (i) throughout this paper, since OPE in (ii) is problematic when the four momentum of the pion becomes soft. The lowest non-trivial order of OPE in (i) is known to give $g_{\pi N}^{lowest} = M_N/f_\pi$ when the continuum threshold is neglected [4,5]. Th origin of the 25% disagreement of $g_{\pi N}^{lowest}$ from the Goldberger-Treiman (GT) relation $g_{\pi N} = g_A M_N/f_\pi$ may originate either from the higher dimensional operators, α_s corrections and the continuum threshold. We will examine $g_{\pi N}$ with all these ingredients.

In section II, we will examine QSR for the nucleon mass by adopting OPE up to dimension 7 operators with $O(\alpha_s)$ corrections, which is essential for the discussions in later sections. In section III, QSR for $g_{\pi N}$ is studied in close analogy with that for the nucleon mass. In section IV, Borel analyses for $g_{\pi N}$ are made, and the effect of the α_s corrections, higher dimensional operators and the continuum thresholds are studied. Section V is devoted to summary and concluding remarks.

II. SUM RULES FOR THE NUCLEON

Let's consider the two point function,

[1] There is a QSR study of the axial charge g_A, which can be related to $g_{\pi N}$ assuming the Goldberger-Treiman (GT) relation [6]. In the present paper, however, we will study $g_{\pi N}$ directly.

$$\Pi_{\alpha\beta}(q) = i\int d^4x e^{iq\cdot x}\langle 0|T\eta_\alpha(x)\bar{\eta}_\beta(0)|0\rangle$$
$$= \Pi_1(q)\hat{q}_{\alpha\beta} + \Pi_2(q)\mathbf{1}_{\alpha\beta}, \tag{1}$$

with $\hat{q} \equiv q\cdot\gamma$. For interpolating nucleon current, we use the Ioffe current [7]

$$\eta(x) = \epsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))\gamma_5\gamma^\mu d^c(x), \tag{2}$$

where a, b and c are color indices and C is the charge conjugation operator.

The operator product expansion (OPE) at $q^2 \to -\infty$ has a general form

$$\int d^4x e^{iq\cdot x} T\eta(x)\bar{\eta}(0) = \sum_n C_n(q,\mu,\alpha_s(\mu^2))O_n(\mu), \tag{3}$$

where C_n denote the Wilson coefficients and O_n are the local gauge invariant operators. They depend on the renormalization scale μ which separates the short distance dynamics in C_n and the long distance dynamics in O_n. If one takes the vacuum expectation value of (3), it can be used for the sum rules of the nucleon, while if one takes the vacuum to pion matrix element, it can be used for the sum rules of $g_{\pi N}$.

The Lorentz structure of O_n depends on the states one chooses to sandwich (3). For the nucleon sum rules, only the scalar operators contribute. OPE up to dimension 7 with α_s corrections in the chiral limit can be extracted from refs. [8] and [9];

$$\mathrm{Re}\Pi(Q^2) = \left(\Pi_1^a + \Pi_1^b + \Pi_1^c\right)\hat{q} + \left(\Pi_2^d + \Pi_2^e + \Pi_2^f\right), \tag{4}$$
$$\Pi_1^a(Q^2) = \frac{-1}{64\pi^4}Q^4 \ln\frac{Q^2}{\mu^2}\left(1 + \frac{71}{12}\frac{\alpha_s}{\pi} - \frac{1}{2}\frac{\alpha_s}{\pi}\ln\frac{Q^2}{\mu^2}\right), \tag{5}$$
$$\Pi_1^b(Q^2) = \frac{-1}{32\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle \ln\frac{Q^2}{\mu^2}, \tag{6}$$
$$\Pi_1^c(Q^2) = \frac{2}{3}\frac{\langle\bar{u}u\rangle^2}{Q^2}\left\{1 - \frac{\alpha_s}{\pi}\left(\frac{1}{3}\ln\frac{Q^2}{\mu^2} + \frac{5}{6}\right)\right\}, \tag{7}$$
$$\Pi_2^d(Q^2) = \frac{-\langle\bar{d}d\rangle}{4\pi^2}Q^2\ln\frac{Q^2}{\mu^2}\left(1 + \frac{3}{2}\frac{\alpha_s}{\pi}\right), \tag{8}$$
$$\Pi_2^e(Q^2) = 0, \tag{9}$$
$$\Pi_2^f(Q^2) = \frac{1}{18Q^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\langle\bar{d}d\rangle, \tag{10}$$

where $Q^2 \equiv -q^2 \to \infty$ and $\langle\cdot\rangle$ denotes the vacuum expectation value. The argument of α_s is μ^2 which is not written explicitly. The diagrammatic illustration of $\Pi_1^a \sim \Pi_2^f$ is shown in Fig. 1.

Several remarks are in order here.

(a) We take the chiral limit ($m_q = 0$) throughout this paper. Small u, d quark mass does not change the essential conclusion of this paper.

(b) The fact that the Wilson coefficient of the dimension 5 operator $g\bar{q}\sigma\cdot Gq$ vanishes in Π_2 is a unique property of the Ioffe current [7]. Since this operator is already $O(g)$, we do not consider the $O(\alpha_s)$ correction to it.
(c) A small discrepancy between the formula in [9] and that in [10] for Π_2^d has been recently resolved (see [11]) and the final result boils down to the form given in the above.
(d) We always assume the vacuum saturation when evaluating matrix elements of higher dimensional operators.

The correlation function (1) satisfies the standard dispersion relation

$$\mathrm{Re}\Pi_{1,2}(Q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\Pi_{1,2}(s)}{s+Q^2}ds + \mathrm{subtraction}. \tag{11}$$

In QSR, $\mathrm{Re}\Pi(Q^2)$ in the left hand side of (11) is calculated by OPE at large Q^2 as give in (4), while $\mathrm{Im}\Pi(s)$ in the right hand side is parametrized by the nucleon pole and the phenomenological continuum. The pole part reads

$$\hat{q}\mathrm{Im}\Pi_1^{pole}(s) + \mathrm{Im}\Pi_2^{pole}(s) = \pi\lambda_N^2(\hat{q}+M_N)\delta(s-M_N^2), \tag{12}$$

where λ_N is defined as $\langle 0|\eta|N\rangle = \lambda_N u(p)$ with $u(p)$ being the nucleon Dirac spinor. We assume that, when $s > S_N$, the hadronic continuum reduces to the same form with that obtained by an analytic continuation of OPE;

$$\mathrm{Im}\Pi_1^{cont}(s) \;=\; \pi\theta(s-S_N)\left[\frac{s^2}{64\pi^4}\left\{1+\frac{\alpha_s}{\pi}\left(\frac{71}{12}-\log(\frac{s}{\mu^2})\right)\right\}+\frac{1}{32\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\right], \tag{13}$$

$$\mathrm{Im}\Pi_2^{cont}(s) = -\pi\theta(s-S_N)\frac{\langle\bar{d}d\rangle}{4\pi^2}s\left(1+\frac{3}{2}\frac{\alpha_s}{\pi}\right). \tag{14}$$

Substituting (4) and (12,13,14) into the dispersion relation (11), one can generate sum rules for the resonance parameters. We will write here the Borel transformed version of the sum rules in which the higher dimensional operators and the effect of the continuum can be relatively suppressed. (See Appendix A for useful formula.)

$$\begin{aligned} 4\pi^4\lambda_N^2 e^{-M_N^2/M^2} &= \frac{M^6}{8}E_2(x)\left(1+(\frac{53}{12}+\gamma_E)\frac{\alpha_s(M^2)}{\pi}\right) \\ &\quad +\frac{bM^2}{32}E_0(x)+\frac{a_u^2}{6}\left(1-(\frac{5}{6}-\frac{1}{3}\gamma_E)\frac{\alpha_s(M^2)}{\pi}\right), \end{aligned} \tag{15}$$

$$4\pi^4\lambda_N^2 M_N e^{-M_N^2/M^2} = \frac{a_d}{4}M^4E_1(x)\left(1+\frac{3}{2}\frac{\alpha_s(M^2)}{\pi}\right)-\frac{a_d b}{72}. \tag{16}$$

Here M^2 is the Borel mass, $E_n = 1-(1+x+\cdots+\frac{x^n}{n!})e^{-x}$ with $x = S_N/M^2$, γ_E is the Euler constant $(0.5772\cdots)$ and

$$a_q \equiv -4\pi^2\langle\bar{q}q\rangle, \qquad b \equiv 4\pi^2\langle\frac{\alpha}{\pi}G^2\rangle. \tag{17}$$

Note that we chose μ^2 to be M^2 which is a typical scale of the system after the Borel transform. We call eq. (15) (eq. (16)) as "even" ("odd") sum rule since it contains only even (odd) dimensional operators. As for the running coupling constant, we use a simplest one-loop form,

$$\alpha_s(M^2) = \frac{4\pi}{9\log(M^2/\Lambda^2)}, \tag{18}$$

with $\Lambda^2 = (0.174GeV)^2$ which is obtained to reproduce $\alpha_s(1) \simeq 0.4$ [12].

Eq.(15) and Eq. (16) will be used when we analyse the sum rules for $g_{\pi N}$. One can derive formula for M_N as a function of M in three ways; (i) the ratio of (15) and (16), (ii) the ratio of (15) and its logarithmic derivative with respect to M^2 and (iii) the ratio of (16) and its logarithmic derivative with respect to M^2. We have made an extensive Borel stability analyses for these three cases. We found that the higher dimensional operator and the α_s corrections improve the Borel stability as well as the prediction for M_N. However, there is a wide range of S_N which can reproduce the experimental nucleon mass in the Borel analyses. In later sections, we will use the finite energy sum rules (FESR) to fix S_N.

In Fig.1, shown are the Borel curves for three cases (i)-(iii) with $S_N = 1.601GeV^2$ obtained by the FESR with $\langle\bar{q}q\rangle = -(0.2402GeV)^3$ and $\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.012GeV^4$. The solid, dashed and dash-dotted curves correspond to cases (i), (ii) and (iii), respectively. The curve in case (iii) is the most stable of three cases and reproduces the experimental value. The others do not show good stability. Also, three curves are rather sensitive to the change of S_N.

III. SUM RULES FOR THE $\pi - N$ COUPLING CONSTANT

In this section, we look at the two-point correlation function,

$$\Pi^\pi_{\alpha\beta}(q) = i\int d^4x e^{iq\cdot x}\langle 0|T\eta_\alpha(x)\bar{\eta}_\beta(0)|\pi^0(p=0)\rangle, \tag{19}$$

where $|\pi^0(p=0)\rangle$ is a neutral pion state with vanishing four momentum (the soft pion). Since we are working in the chiral limit, this soft pion is simultaneously on shell.

Since we are taking the matrix element between the vacuum and the soft pion in (19), only the pseudo-scalar operator $\bar{q}i\gamma_5 q$ survives in OPE for $\mathrm{T}\eta(x)\bar{\eta}(0)$. The pseudo-vector operator $\bar{q}\gamma_\mu\gamma_5 q$ vanishes since $\langle 0|\bar{q}\gamma_\mu\gamma_5 q|\pi^0(p)\rangle \propto p_\mu \to 0$. Thus the relevant terms in OPE read as follows:

$$\Pi^{\pi}(Q^2) = \Pi_{dim3}(Q^2) + \Pi_{dim5}(Q^2) + \Pi_{dim7}(Q^2), \tag{20}$$

$$\Pi_{dim3}(Q^2) = -i\gamma_5 \frac{\langle 0|\bar{d}i\gamma_5 d|\pi\rangle}{4\pi^2}\left(1 + \frac{3}{2}\frac{\alpha_s}{\pi}\right) Q^2 \ln\frac{Q^2}{\mu^2}, \tag{21}$$

$$\Pi_{dim5}(Q^2) = 0, \tag{22}$$

$$\Pi_{dim7}(Q^2) = i\gamma_5 \frac{1}{18Q^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\langle 0|\bar{d}i\gamma_5 d|\pi\rangle. \tag{23}$$

μ^2 dependence of α_s is again implicit in the above equations.

The diagrams corresponding to (21), (22) and (23) are Fig.1(d), 1(e) and 1(f) respectively. The Wilson coefficients with $O(\alpha_s)$ corrections in (21), (22) and (23) turn out to be identical to (8), (9) and (10) respectively. This can be explicitly checked by carrying out OPE with the background field method in the fixed point gauge. An alternative way to see this is the plain wave method. As an illustration, let's consider dimension 3 operator (Fig.1(d)) and expand eq.(1) in terms of $\bar{d}d$ and $\bar{d}i\gamma_5 d$;

$$i\int d^4x e^{iq\cdot x} T(\eta_\alpha(x)\bar{\eta}_\beta(0)) = C_s(1)_{\alpha\beta}\bar{d}d + C_{ps}(i\gamma_5)_{\alpha\beta}\bar{d}i\gamma_5 d + \cdots \tag{24}$$

where $C_{s,ps}$ is the Wilson coefficient with $s(ps)$ denoting scalar (pseudoscalar). Note that $\cdots$ are vector, pseudovector and tensor operators. Note also that $\bar{u}u$ and $\bar{u}i\gamma_5 u$ do not arise for the Ioffe current. We sandwich eq.(24) by free quark states to extract the Wilson coefficients. Applying the projections $1_{\alpha\beta}$ and $(i\gamma_5)_{\alpha\beta}$ to eq.(24), one gets

$$C_s = \frac{L_1(q^2 \to -\infty)}{\langle p|\bar{d}d|p\rangle}, \qquad L_1 = \frac{i}{4}\int d^4x e^{iq\cdot x}\langle p|T(\eta(x)\bar{\eta}(0))|p\rangle \tag{25}$$

$$C_{ps} = \frac{L_{i\gamma_5}(q^2 \to -\infty)}{\langle p|\bar{d}i\gamma_5 d|p\rangle}, \qquad L_{i\gamma_5} = \frac{-i}{4}\int d^4x e^{iq\cdot x}\langle p|T(\eta(x)i\gamma_5\bar{\eta}(0))|p\rangle \tag{26}$$

with $|p\rangle$ being a quark state with momentum p. Since massless QCD does not flip chirality in perturbation theory, $\langle p|\bar{d}d|p\rangle \sim \langle p|\bar{d}i\gamma_5 d|p\rangle$ and $L_1 \sim L_{i\gamma_5}$ except for trivial kinematical factors. Thus $C_s = C_{ps}$ is obtained even when α_s corrections are included.

$\Pi^{\pi}(Q^2)$ satisfies the dispersion relation

$$\mathrm{Re}\Pi^{\pi}(Q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\Pi^{\pi}(s)}{s + Q^2}ds + \text{subtraction}. \tag{27}$$

The hadronic imaginary part $\mathrm{Im}\Pi^{\pi}(s)$ has a nucleon pole and the continuum. The pole part is parametrized by assuming the $\pi - N - N$ vertex $\mathcal{L}_{int} = g_{\pi N}\bar{N}i\gamma_5\vec{\tau}\cdot\vec{\pi}N$ as was done in [4,5]. The continuum part is extracted from the analytic continuation of OPE. In total,

$$\mathrm{Im}\Pi^\pi(s) = i\gamma_5\pi\lambda_N^2 g_{\pi N}\delta(s-M_N^2) \\ -i\gamma_5\pi\frac{s}{4\pi^2}\left(1+\frac{3}{2}\frac{\alpha_s}{\pi}\right)\theta(s-S_\pi)\langle 0|\bar{d}i\gamma_5 d|\pi^0\rangle, \tag{28}$$

where S_π is the continuum threshold. Note that S_π does not have to be the same with S_N.

Putting Eq.(28) and Eq.(20) into (27) and making the Borel improvement, one obtains the following sum rule:

$$g_{\pi N} = \frac{e^{\frac{M_N^2}{M^2}}}{\lambda_N^2}\left\{\frac{M^4}{4\pi^2}E_1(\frac{S_\pi}{M^2})\left(1+\frac{3}{2}\frac{\alpha_s}{\pi}\right) - \frac{1}{18}\langle\frac{\alpha_s}{\pi}G^2\rangle\right\}\left(-\frac{1}{f_\pi}\langle\bar{d}d\rangle\right), \tag{29}$$

where f_π is the pion decay constant (93 MeV), and $\langle 0|\bar{d}i\gamma_5 d|\pi^0\rangle$ has been rewritten by the soft pion theorem

$$\langle 0|\bar{d}i\gamma_5 d|\pi^0(p=0)\rangle = \frac{1}{f_\pi}\langle\bar{d}d\rangle. \tag{30}$$

One can get rid of the coefficient $e^{M_N^2/M^2}/\lambda_N^2$ in (29) by using the nucleon sum rules (15) or (16). Thus one arrives at two different sum rules for $g_{\pi N}$:

$$g_{\pi N}^{even} = \frac{\pi^2\left\{M^4\left(1+\frac{3}{2}\frac{\alpha_s(M^2)}{\pi}\right)E_1(\frac{S_\pi}{M^2}) - \frac{b}{18}\right\}\left(-\frac{1}{f_\pi}\langle\bar{d}d\rangle\right)}{\frac{M^6}{8}E_2(\frac{S_N}{M^2})\left(1+(\frac{53}{12}+\gamma_E)\frac{\alpha_s(M^2)}{\pi}\right) + \frac{bM^2}{32}E_0(\frac{S_N}{M^2}) + \frac{a_u^2}{6}\left(1-(\frac{5}{6}-\frac{1}{3}\gamma_E)\frac{\alpha_s(M^2)}{\pi}\right)}, \tag{31}$$

which is obtained from the "even" sum rule (15) for the nucleon, and

$$g_{\pi N}^{odd} = \frac{M_N\left\{E_1(\frac{S_\pi}{M^2})\left(1+\frac{3}{2}\frac{\alpha_s(M^2)}{\pi}\right) - \frac{b}{18M^4}\right\}}{f_\pi\left\{E_1(\frac{S_N}{M^2})\left(1+\frac{3}{2}\frac{\alpha_s(M^2)}{\pi}\right) - \frac{b}{18M^4}\right\}}, \tag{32}$$

which is obtained from the "odd" sum rule (16) for the nucleon. For M_N in eq.(32), we just take the experimental number instead of reexpressing M_N by the Borel mass M through the nucleon sum rule. Even if one uses $M_N(M)$ in eq.(32), the Borel curve for $g_{\pi N}^{odd}$ is not affected so much as far as one adopts the odd nucleon sum rule (case (iii) in section II). Note here that, if one assumes $S_\pi = S_N$, Eq.(32) gives $g_{\pi N}^{odd} = M_N/f_\pi$, namely the GT relation with $g_A = 1$.

It is in order here to remark some difference of the present work from that of ref. [4,5]. First of all, we have carried out OPE up to dimension 7 both for $g_{\pi N}$ and the nucleon mass, while only the lowest dimension operator, namely $\bar{q}i\gamma_5 q$, is taken into account for the $g_{\pi N}$ sum rules in ref. [4,5]. Secondly, α_s corrections to the Wilson coefficients are not taken into account in [4,5]. Thirdly, the continuum thresholds S_π and S_N are completely neglected in [4,5] for the $g_{\pi N}$ sum rules.

IV. BOREL ANALYSIS FOR THE $\pi - N$ COUPLING CONSTANT

A. The determination of the thresholds S_π and S_N

As we have mentioned in section II, it is difficult to determine S_N from the Borel sum rules of the nucleon. In this section, we will utilize the finite energy sum rules (FESR) to give a constraint on S_N as well as S_π.

The FESR has a general form [13] ,

$$\int_0^{S_0} ds\ s^n\ \mathrm{Im}\Pi_{OPE}(s) = \int_0^{S_0} ds\ s^n\ \mathrm{Im}\Pi_{phen.}(s), \tag{33}$$

where $\mathrm{Im}\Pi_{OPE}(s)$ is an imaginary part obtained by the analytic continuation of Π in OPE, $\mathrm{Im}\Pi_{phen.}(s)$ is the phenomenological imaginary part and S_0 is the continuum threshold (either S_N or S_π). For the nucleon, by using Π_1 with $n = 0, 1$ and Π_2 with $n = 0$, one gets [9]

$$64\pi^4\lambda_N^2 = \left(1 + \frac{25}{4}\frac{\alpha_s}{\pi}\right)\frac{S_N^3}{3} + 2\pi^2\langle\frac{\alpha_s}{\pi}G^2\rangle S_N + \frac{128}{3}\pi^4\langle\bar{q}q\rangle^2\left(1 - \frac{5}{6}\frac{\alpha_s}{\pi}\right), \tag{34}$$

$$64\pi^4\lambda_N^2 M_N = -8\pi^2\langle\bar{q}q\rangle\left(1 + \frac{3}{2}\frac{\alpha_s}{\pi}\right)S_\pi^2 + \frac{32}{9}\pi^4\langle\bar{q}q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle, \tag{35}$$

$$64\pi^4\lambda_N^2 M_N^2 = \left(1 + \frac{37}{6}\frac{\alpha_s}{\pi}\right)\frac{S_N^4}{4} + \pi^2\langle\frac{\alpha_s}{\pi}G^2\rangle S_N^2 - \frac{128}{9}\pi^4\langle\bar{q}q\rangle^2\frac{\alpha_s}{\pi}S_N. \tag{36}$$

The renormalization point here is chosen as $\mu^2 = S_N$.

By using the standard values of the condensates $\langle\bar{q}q\rangle(1\mathrm{GeV}^2) = -(225 \pm 25MeV)^3$ and $\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.012GeV^4$, we solved Eq.(34) $\sim$ Eq.(36) numerically. Table 1 shows the results for four different values of $\langle\bar{q}q\rangle$.

$\langle\bar{q}q\rangle$	$(-0.250GeV)^3$	$(-0.240GeV)^3$	$(-0.225GeV)^3$	$(-0.200GeV)^3$
$S_N(GeV^2)$	1.77	1.60	1.34	0.887
$\lambda_N(GeV^3)$	0.0267	0.0235	0.0187	0.0118
$M_N(GeV)$	0.997	0.940	0.845	0.645

Table1: S_N, λ_N, M_N obtained from Eq.(34) $\sim$ (36) with four different values of $\langle\bar{q}q\rangle$. $\langle\bar{q}q\rangle = -(0.2402GeV)^3$ reproduces the nucleon mass.

From this table, we choose $S_N = 1.34 - 1.77$ as physical range where the nucleon mass is reasonable reproduced within $\pm 10\%$ errors. Our S_N is smaller than that usually used in the literatures [6–8,13]. However, the α_s corrections are not

taken into account in these references. The effect of the α_s correction to the spectral parameters can be explicitly seen by expanding the solutions of Eq.(34) up to linear in α_s;

$$\lambda_N^2 = \lambda_0^2 \left(1 - 26.2(-16\pi^2\langle\bar{q}q\rangle)^{-\frac{4}{3}}\langle\frac{\alpha_s}{\pi}G^2\rangle\right)(1 - 3.08\frac{\alpha_s}{\pi}), \tag{37}$$

$$S_N = S_N^0 \left(1 - 21.2(-16\pi^2\langle\bar{q}q\rangle)^{-\frac{4}{3}}\langle\frac{\alpha_s}{\pi}G^2\rangle\right)(1 - 3.26\frac{\alpha_s}{\pi}), \tag{38}$$

$$M_N = M_N^0 \left(1 - 18.6(-16\pi^2\langle\bar{q}q\rangle)^{-\frac{4}{3}}\langle\frac{\alpha_s}{\pi}G^2\rangle\right)(1 - 1.94\frac{\alpha_s}{\pi}), \tag{39}$$

where $\lambda_0^2 = 4\langle\bar{q}q\rangle^2$, $S_N^0 = (640\pi^4\langle\bar{q}q\rangle^2)^{\frac{1}{3}}$ and $M_N^0 = (-\frac{25}{2}\pi^2\langle\bar{q}q\rangle)^{\frac{1}{3}}$, which are the solutions when the α_s corrections and the gluon condensate are neglected. (37)-(39) show that the α_s corrections tend to reduce the observables by considerable amount particularly in S_N.

Next, we estimate S_π by taking the $n = 0$ FESR of Π^π;

$$\lambda_N^2 g_{\pi N} = \left\{\frac{1}{8\pi^2}S_\pi^2\left(1 + \frac{3}{2}\frac{\alpha_s(S_\pi)}{\pi}\right) - \frac{1}{18}\langle\frac{\alpha_s}{\pi}G^2\rangle\right\}\left(-\frac{1}{f_\pi}\langle\bar{d}d\rangle\right). \tag{40}$$

Since the FESR is rather sensitive to the structure of the continuum compared to the Borel sum rule and we do not know much about the detailed structure of the continuum for (19), we just limit ourselves to the $n = 0$ sum rule (local duality relation) for safety. To roughly evaluate the range of S_π, we simply put $g_{\pi N} = 13.4$ in (40) with λ_N being determined in the nucleon FESR. The result is given in Table 2 for three different values of the condensate:

	$\langle\bar{q}q\rangle(GeV^3)$	$S_N(GeV^2)$	$S_\pi(GeV^2)$
set 1	$-(0.250)^3$	1.77	1.98
set 2	$-(0.240)^3$	1.60	1.85
set 3	$-(0.225)^3$	1.34	1.62

Table 2: S_N, S_π with three different values of $\langle\bar{q}q\rangle$. S_π is obtained by substituting g_π=13.4 into (40).

From Table 2, one finds that S_π is always greater than S_N. This is consistent with the Borel sum rule $g_{\pi N}^{odd}$ in (32) which tells us that $g_{\pi N} > M_N/f_\pi$ only when $S_\pi > S_N$. In subsections below, we will examine the Borel stability of $g_{\pi N}$ with the parameter sets obtained in Table 2.

B. Borel analysis for $g_{\pi N}^{even}$

In Fig.3, $g_{\pi N}^{even}$ is shown as a function of M^2. The solid, dashed and dash-dotted curves correspond to set 1, set 2 and set 3 in Table 2, respectively. $g_{\pi N}^{even}$ in Fig.3(a) includes the α_s corrections to the Wilson coefficients, while they are neglected in Fig.3(b) except for the gluon condensate.

$g_{\pi N}^{even}$ has a sizable M^2 variation and good Borel stability is not seen in Fig.3. By comparing Fig.3(a) with Fig.3(b), one finds that the α_s corrections improve the Borel stability only slightly. The effect of the higher dimensional operator to the Borel curve is also small. In fact, $\frac{2\pi^2}{9M^4}\langle\frac{\alpha_s}{\pi}G^2\rangle/\{E_1(\frac{S_\pi}{M^2})(1+\frac{3}{2}\frac{\alpha_s}{\pi})\}$, which is a ratio of the dimension 3 term and the dimension 7 term in eq.(29), is about 4 % at $M^2 \sim 1\mathrm{GeV}^2$. (Note that dimension 5 terms do not arise for the Ioffe current.)

Although $g_{\pi N}^{even}$ in eq.(31) is proportional to $\langle\bar{q}q\rangle$, three curves in Fig.3, which correspond to different values of $\langle\bar{q}q\rangle$, almost overlap with each other. This is because the change of $\langle\bar{q}q\rangle$ is compensated by the changes of $S_{\pi,N}$.

In Fig.4, $g_{\pi N}^{even}$ for set 2 is shown with S_π being changed by $\pm 10\%$. The solid, dashed and dash-dotted curves correspond to $S_\pi = 1.85\times 1.1, 1.85$ and 1.85×0.9, respectively. $g_{\pi N}^{even}$ increases as S_π increases, which is consistent with the prediction of FESR in eq.(40).

C. Borel analysis for $g_{\pi N}^{odd}$

Fig.5(a),(b) show $g_{\pi N}^{odd}$ as a function of M^2. The solid, dashed, dash-dotted curves correspond to set 1, set 2, set 3, respectively. $g_{\pi N}^{odd}$ in Fig.5(a) includes α_s corrections to the Wilson coefficients, while they are neglected in Fig.5(b) except for the gluon condensate.

$g_{\pi N}^{odd}$ has apparently better Borel stability than $g_{\pi N}^{even}$, but still sizable M^2 variation is seen. The α_s corrections do not affect the Borel stability much, since the same α_s correction appears both in numerator and denominator in eq.(32).

$g_{\pi N}^{odd}$ is rather sensitive to the change of the parameter sets , in particular the S_π/S_N. To see the effect of $S_{\pi,N}$ on $g_{\pi N}^{odd}$ in more detail, we expand eq.(32) up to $O(\frac{1}{M^2})$:

$$\begin{aligned} g_{\pi N}^{odd} &\simeq \frac{M_N}{f_\pi}\frac{(S_\pi^2-\frac{b}{9})}{(S_N^2-\frac{b}{9})}\left\{1-\frac{2}{3M^2}\left(\frac{S_\pi^3}{S_\pi^2-\frac{b}{9}}-\frac{S_N^3}{S_N^2-\frac{b}{9}}\right)\right\} \\ &\simeq \frac{M_N}{f_\pi}(\frac{S_\pi}{S_N})^2\left\{1-\frac{2}{3M^2}(S_\pi-S_N)\right\} \quad , \end{aligned} \tag{41}$$

where we have neglected small α_s corrections and used the fact $S_{\pi N}^2 \gg b/9$. The approximate formula eq.(41) is in good agreement with the exact one eq.(32) in 10 % for $M^2 > 1.0\text{GeV}^2$.

When $S_\pi > S_N$, M^2 independent term in (41) gives $g_{\pi N}^{odd} = (S_\pi/S_N)^2(M_N/f_\pi)$ which is larger than M_N/f_π. The leading $1/M^2$ correction reduces $g_{\pi N}^{odd}$ slightly. The experimental value for $g_{\pi N} = 13.4$ is obtained when $M^2 = 1.6\text{GeV}^2$ for the parameter set 3.

In Fig.6, $g_{\pi N}^{odd}$ for set 2 is shown with S_π being changed by $\pm 10\%$. The solid, dashed and dash-dotted curves correspond to $S_\pi = 1.85 \times 1.1, 1.85$ and 1.85×0.9, respectively. $g_{\pi N}^{odd}$ increases as S_π increases, which is consistent with the prediction of FESR in eq.(40) and also with the approximate formula (41).

D. Comparison of $g_{\pi N}^{even}$ and $g_{\pi N}^{odd}$

As we have already mentioned, the Borel stability for $g_{\pi N}^{odd}$ is better than $g_{\pi N}^{even}$. This is consistent with the fact that the Borel curve for M_N is most stable in "odd" sum rule (case (iii) in Fig.2). Also the absolute value of $g_{\pi N}^{odd}$ is larger than $g_{\pi N}^{even}$ for appropriate range of M^2: e.g. $g_{\pi N}^{odd}$ =11.3 − 12.8 versus $g_{\pi N}^{even}$ =9.02 − 9.10 at $M^2 = 1\text{GeV}^2$.

In previous subsections, $S_\pi = 1.85 GeV^2$ has been used as a standard value to study the Borel stability. Alternative way to make the Borel analysis is to eliminate S_π from eq.(31) and eq.(32) using eq.(40), and then to solve $g_{\pi N}$ self-consistently for each M^2. By this procedure, we found that there exists no solution satisfying eq.(31) and eq.(40) simultaneously, while there exists solutions of eq.(32) and eq.(40) which are given in Fig.7. This result again confirms that $g_{\pi N}^{odd}$ is better starting point to study the $\pi - N$ coupling constant than $g_{\pi N}^{even}$.

V. CONCLUSION

In this article we have made extensive Borel and FESR analyses of $g_{\pi N}$ by taking into account higher dimensional operators, α_s corrections and the continuum threshold. None of them has been considered in the previous analyses which led to $g_{\pi N} = M_N/f_\pi$ [4,5].

What we have found are summarized as follows:

(a) The higher dimensional operators up to dim. 7 and the α_s corrections play no crucial role for the Borel stability of $g_{\pi N}$.

(b) $g_{\pi N}^{odd}$ is more appropriate for examining $g_{\pi N}$ than $g_{\pi N}^{even}$, since the former has better

Borel stability. This fact is also consistent with the fact that "odd" sum rule forM_N has a best stability.
(c) $g_{\pi N}$ is most sensitive to the ratio S_π/S_N, and both the FESR and Borel sum rules tell us that $S_\pi/S_N > 1$ is a crucial ingredient to reproduce the experimental $g_{\pi N}$.[2]

We also found that the Borel stability of $g_{\pi N}^{odd}$, even if dim. 7 operator is taken into account, is not satisfactory enough to determine the pion-nucleon coupling constant precisely. For the nucleon sum rule, there have been some attempts to improve the Borel stability such as the modification of the Ioffe current [14] and the inclusion of the instantons [15]. In particular, instantons improve the nucleon Borel sum rules considerably at low M^2 region, so it will be an interesting problem to study $g_{\pi N}$ with instanton contribution in the future.

ACKNOWLEDGMENTS

This work was supported in part by the Grants-in-Aid of the Ministry of Education (No. 06102004). T. H. thanks Institute for Nuclear Theory at the University of Washington for its hospitality and the Department of Energy for partial support during the completion of this work.

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[2]This point may have some relation to the Adler-Weisberger sum rule [16] which tells us that $g_A > 1$ (or equivalently $g_{\pi N} > M_N/f_\pi$) is obtained only when the continuum contribution in the $\pi - N$ channel is taken into account.

APPENDIX A:

The Borel transform is defined as,

$$\hat{B} = \frac{(-1)^n (Q^2)^n}{(n-1)!} \left(\frac{d}{dQ^2} \right)^n , \qquad Q^2 = -q^2 \quad , \tag{A.1}$$

with $Q^2 \to \infty, n \to \infty$, and $\frac{Q^2}{n} = M^2$ being fixed.

When $\hat{B}$ is applied to the correlation function $\Pi(Q^2) = \frac{1}{\pi} \int \frac{\mathrm{Im}\Pi(s)}{s+Q^2}$, it leads to

$$\hat{B}\Pi(Q^2) = \frac{1}{\pi M^2} \int \, ds \; \mathrm{Im}\Pi(s) e^{-\frac{s}{M^2}} \tag{A.2}$$

This shows that the Borel transform tends to suppress the high energy contribution.

Some useful formula are

$$\hat{B} \left(\frac{1}{Q^2} \right)^k = \frac{1}{(k-1)!} \left(\frac{1}{M^2} \right)^k , \tag{A.3}$$

$$\hat{B}(Q^2)^k \log Q^2 = (-1)^{k+1} \Gamma(k+1) (M^2)^k , \tag{A.4}$$

$$\hat{B} \frac{1}{s+Q^2} = \frac{1}{M^2} e^{-\frac{s}{M^2}} , \tag{A.5}$$

$$\hat{B} \frac{\log Q^2}{Q^2} = \frac{1}{M^2} (\log M^2 - \gamma_E) , \tag{A.6}$$

$$\hat{B} (\log Q^2)^2 = -2 \log M^2 + 2\gamma_E , \tag{A.7}$$

$$\hat{B} Q^2 (\log Q^2)^2 = 2M^2 (\log M^2 - \gamma_E + 1) , \tag{A.8}$$

$$\hat{B} (Q^2)^2 (\log Q^2)^2 = (M^2)^2 (-4 \log M^2 + 4\gamma_E - 6) . \tag{A.9}$$

Figure Captions

Fig.1

OPE up to dimension 7 operators for the correlation of Ioffe current. Wavy lines denote gluon lines, broken lines denote the quark/gluon condensate.

Fig.2

M_N (nucleon mass) as a function of the Borel mass squared M^2. The solid, dashed, dash-dotted lines correspond to the cases (i), (ii) and (iii), respectively. $\langle \bar{q}q \rangle = -(0.240GeV)^3$, $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.012GeV^4$ and $S_N = 1.60GeV^2$ are used.

Fig.3(a),(b)

$g_{\pi N}^{even}$ as a function of M^2. The solid, dashed and dash-dotted lines correspond to set 1, set 2 and set 3, respectively. α_s corrections are taken into account in Fig.3(a), while they are neglected in Fig.3(b) except for gluon condensate.

Fig.4

$g_{\pi N}^{even}$ with $\langle \bar{q}q \rangle = -(0.240GeV)^3$ as a function of M^2. The solid, dashed and dash-dotted lines correspond to $S_\pi = 1.85 \times 1.1, 1.85$ and 1.85×0.9, respectively.

Fig.5(a),(b)

$g_{\pi N}^{odd}$ as a function of M^2. The solid, dashed and dash-dotted lines correspond to set 1, set 2 and set 3, respectively. α_s corrections are taken into account in Fig.5(a), while they are neglected in Fig.5(b) except for gluon condensate.

Fig.6

$g_{\pi N}^{odd}$ with $\langle \bar{q}q \rangle = -(0.240GeV)^3$ as a function of M^2. The solid, dashed and dash-dotted lines correspond to $S_\pi = 1.85 \times 1.1, 1.85$ and 1.85×0.9, respectively.

Fig.7

$g_{\pi N}$ as a function of M^2. S_π in eq.(40) with λ_N being determined in the nucleon FESR is used for S_π in eq.(32). The solid, dashed and dash-dotted lines correspond to $g_{\pi N}^{odd}$ with $\langle \bar{q}q \rangle = -(0.25GeV)^3, -(0.240GeV)^3$ and $-(0.225GeV)^3$, respectively.

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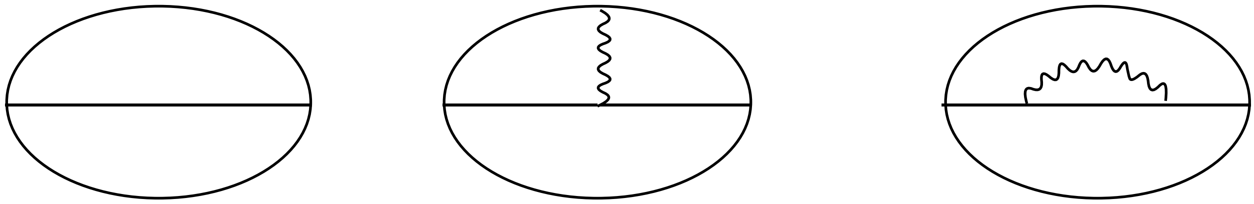
a)



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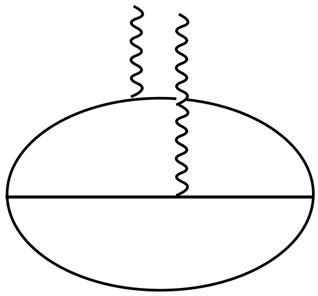

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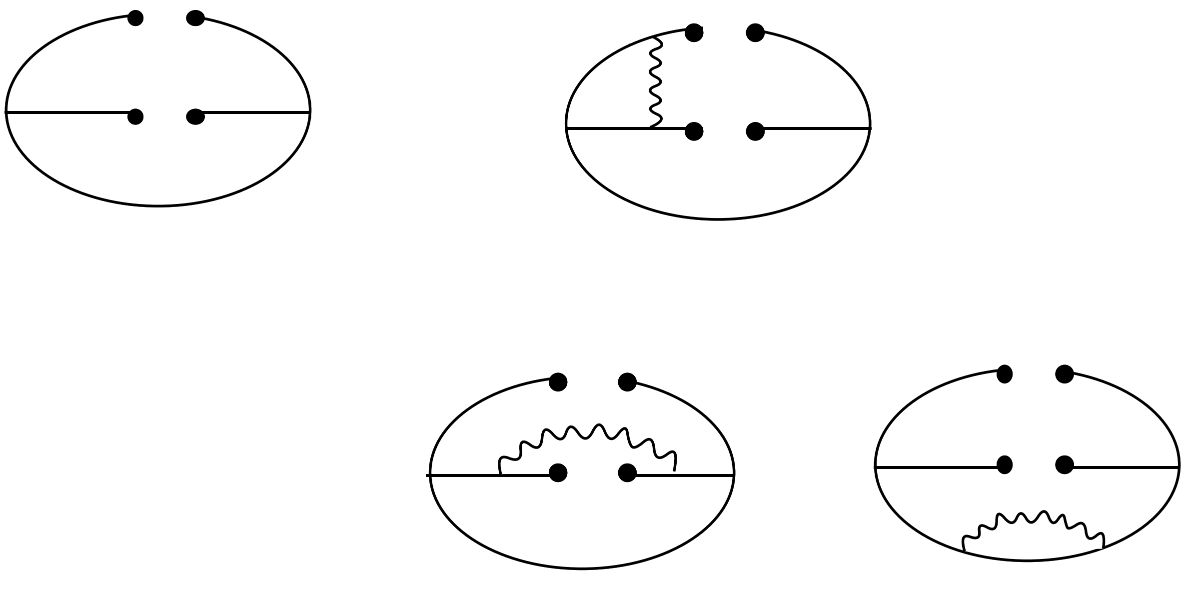


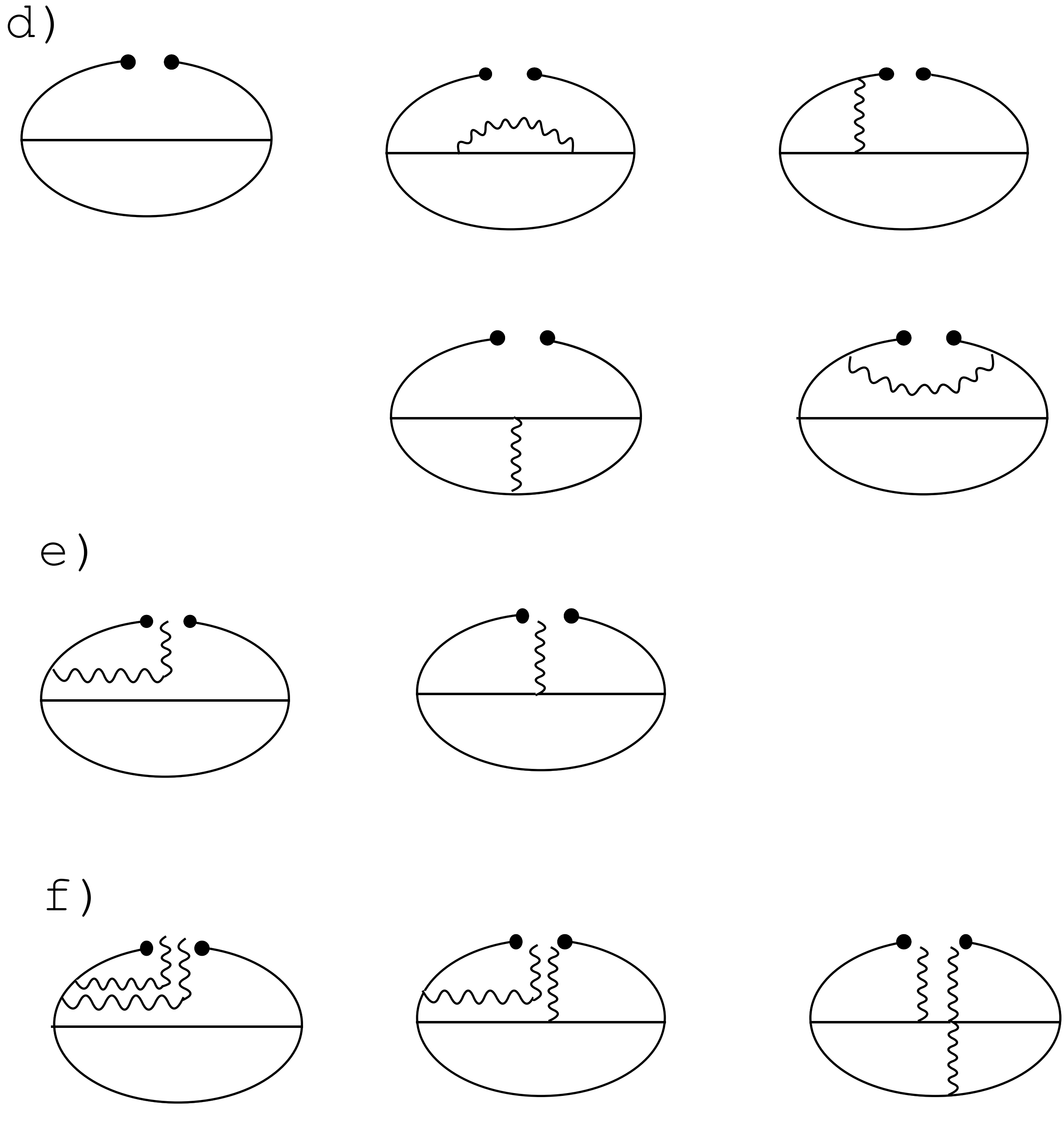
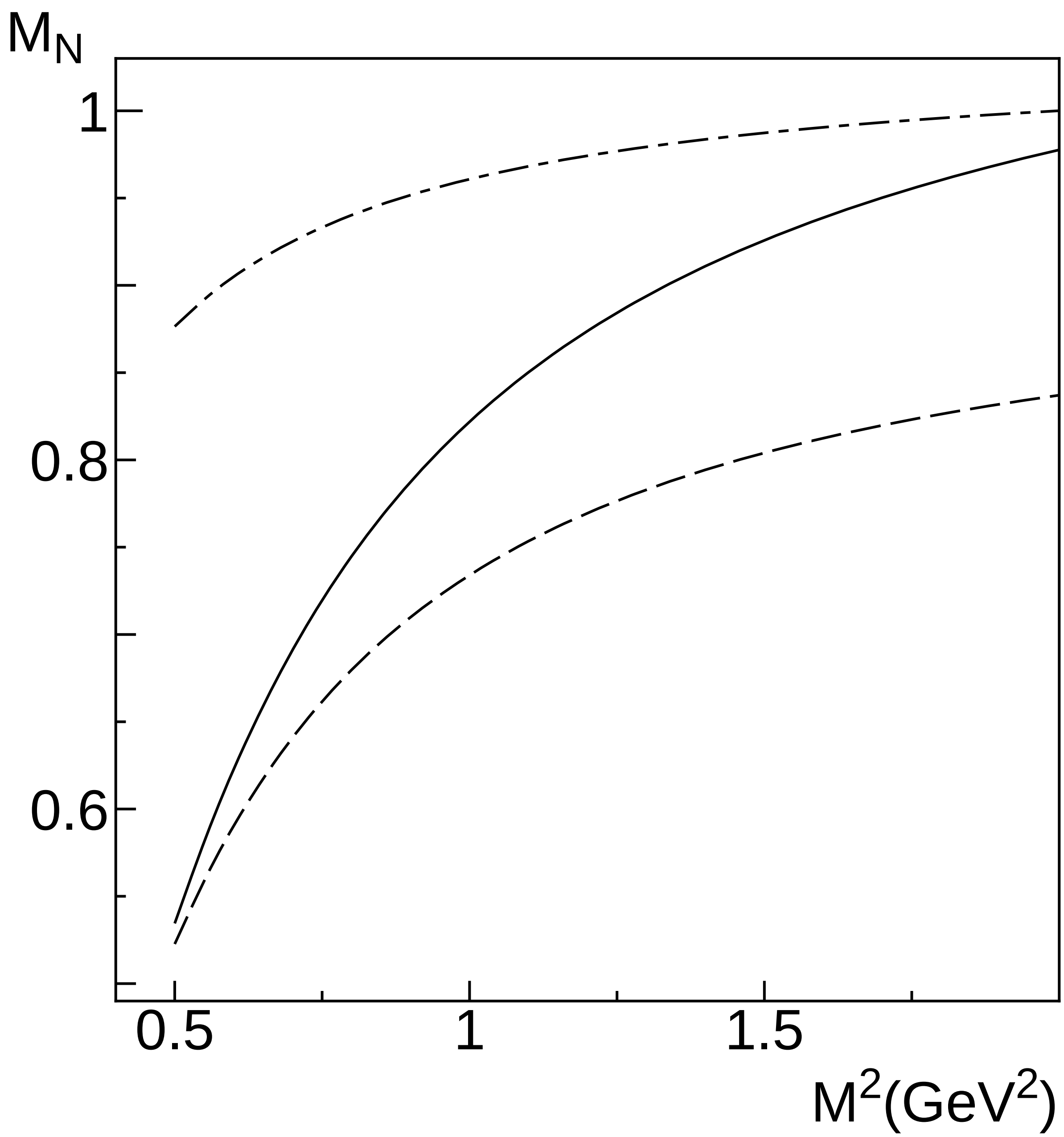


Fig.1



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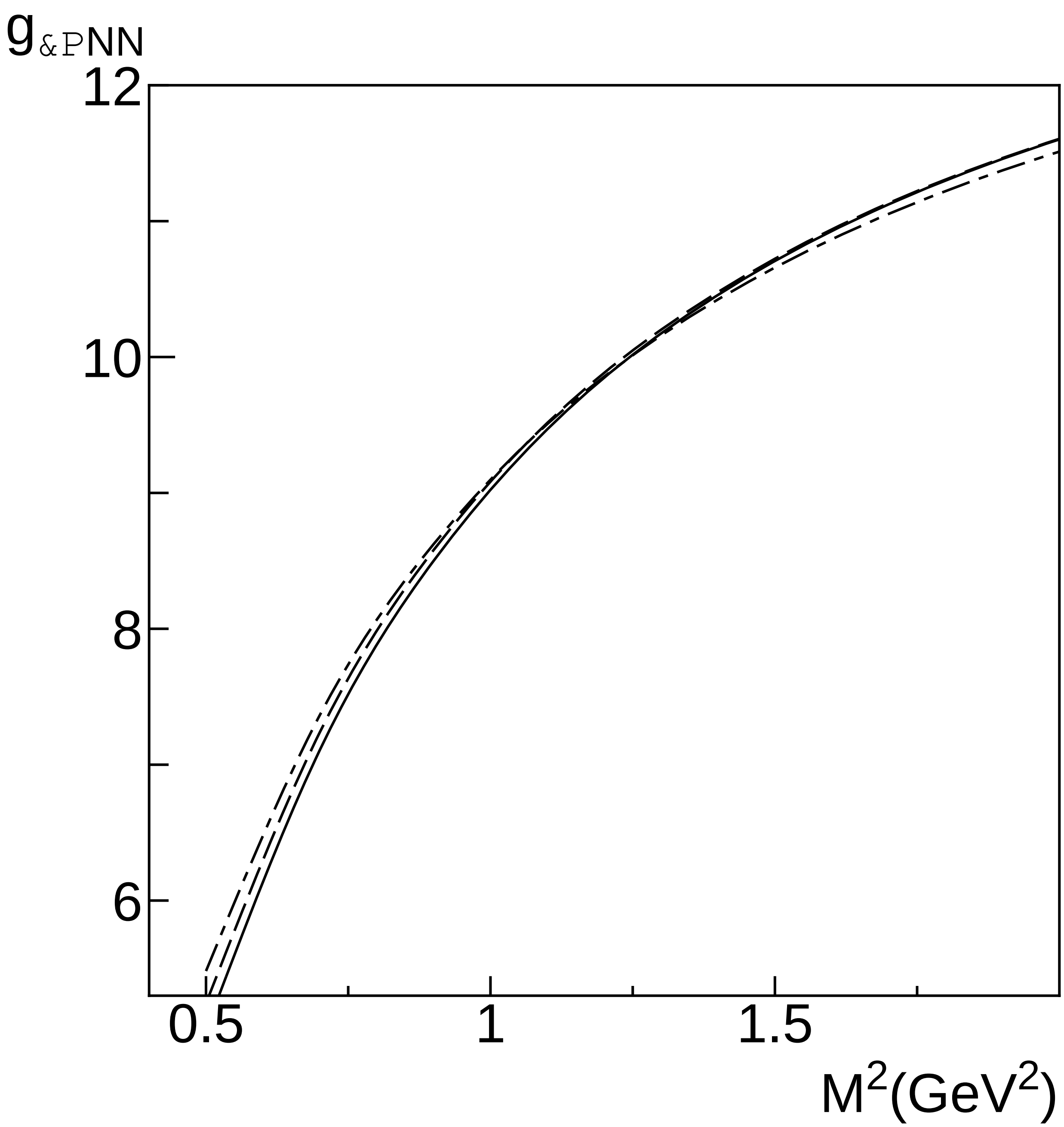


Fig.3(a)

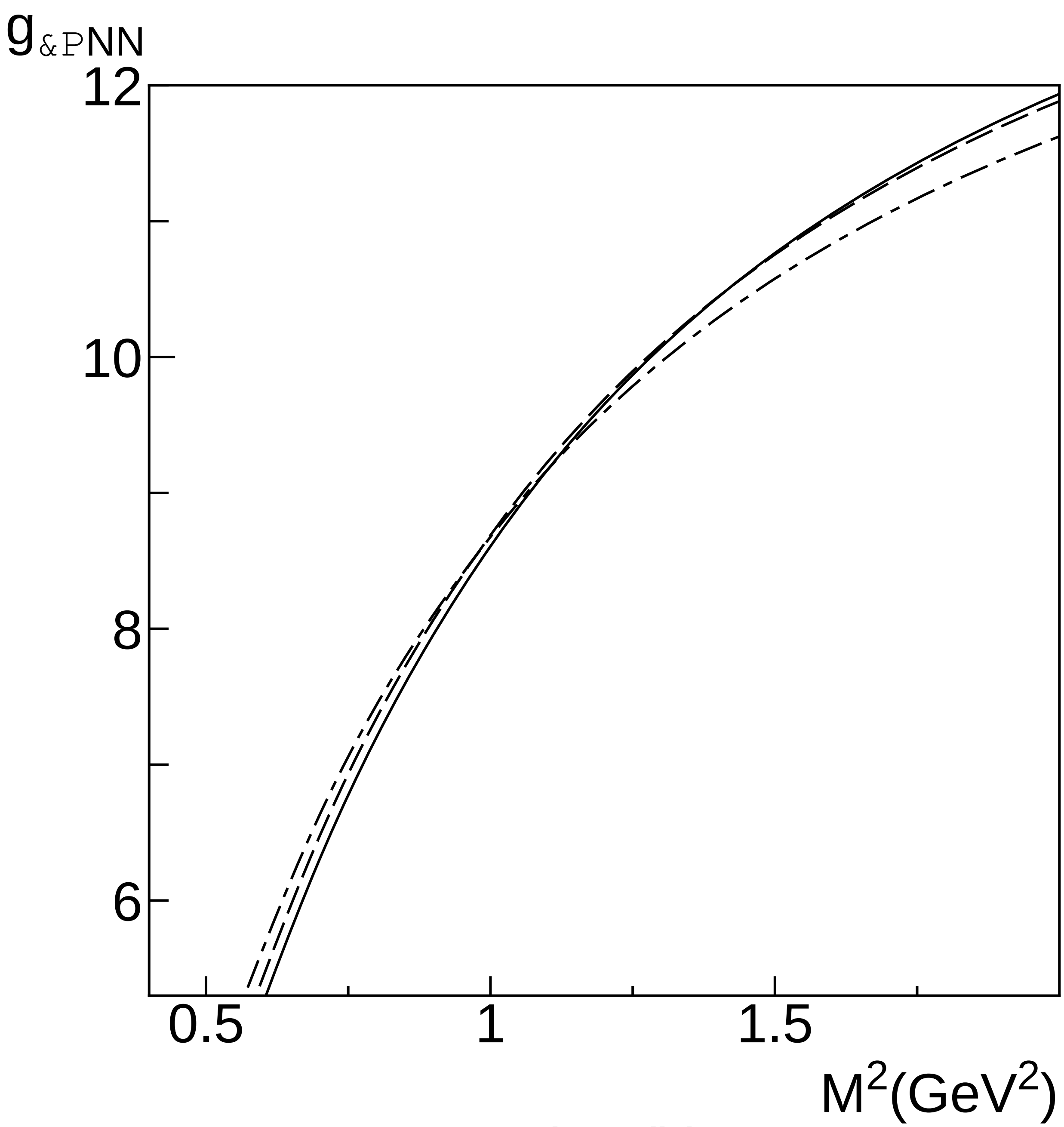


Fig.3(b)

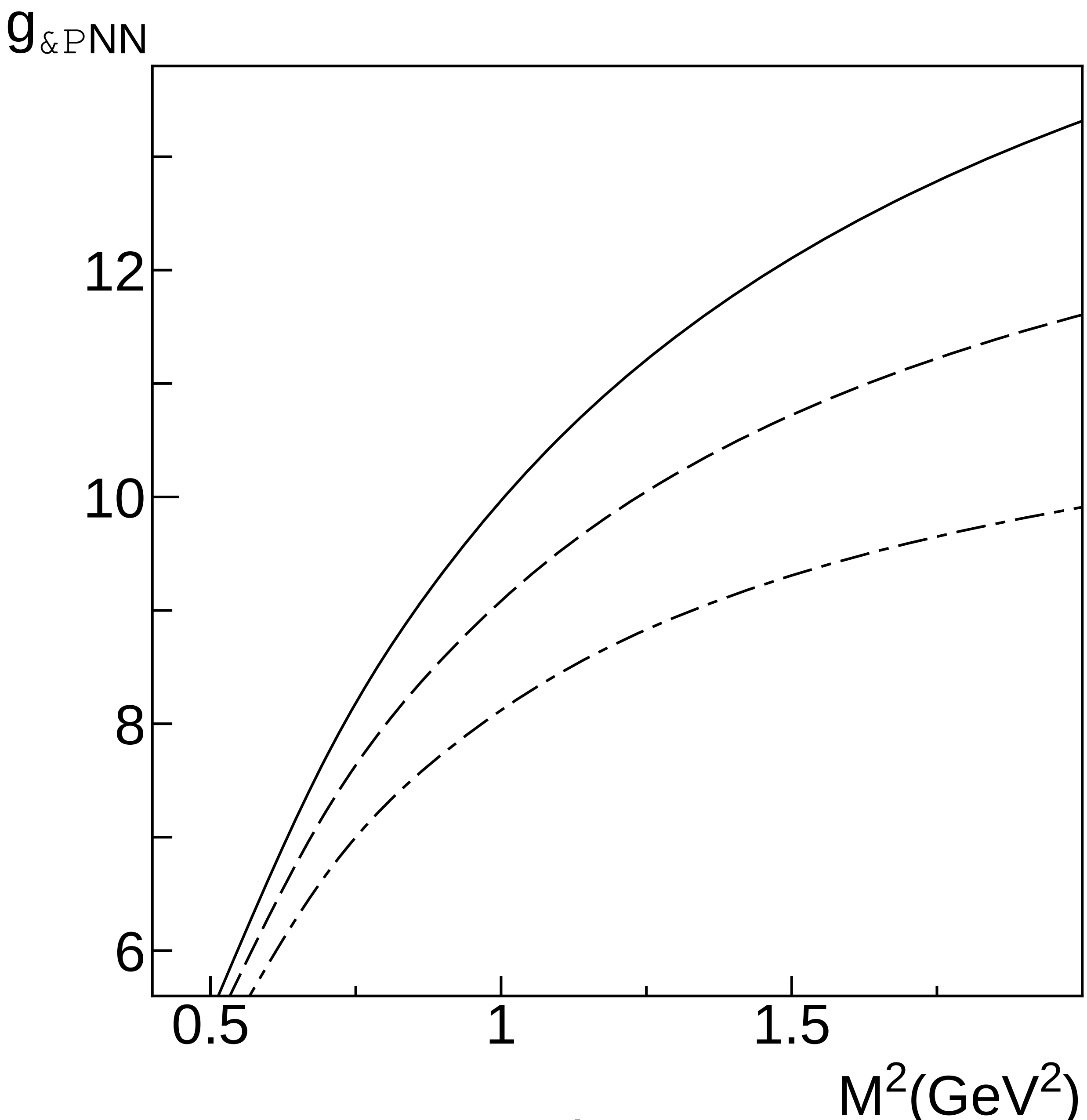


Fig.4

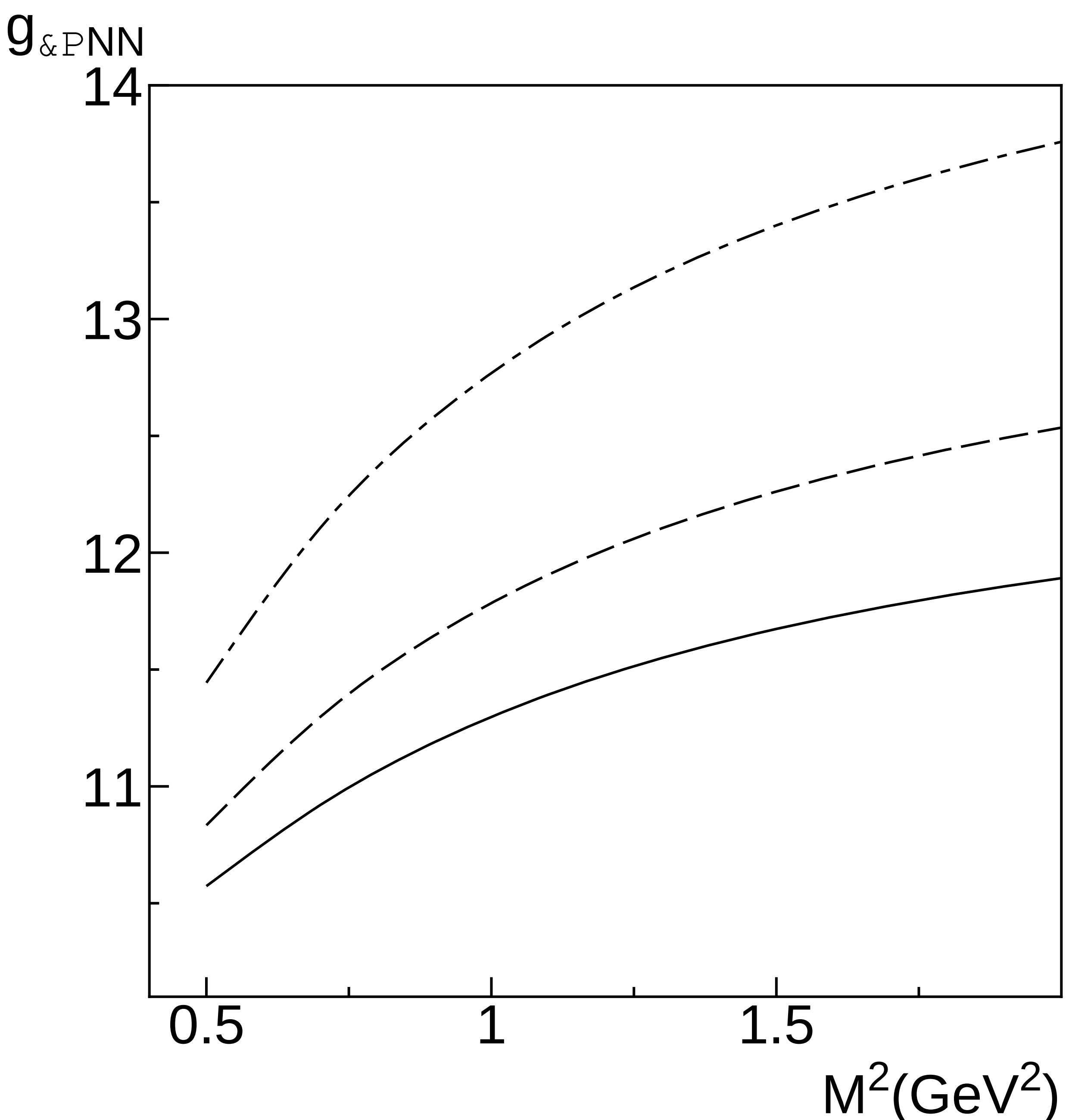
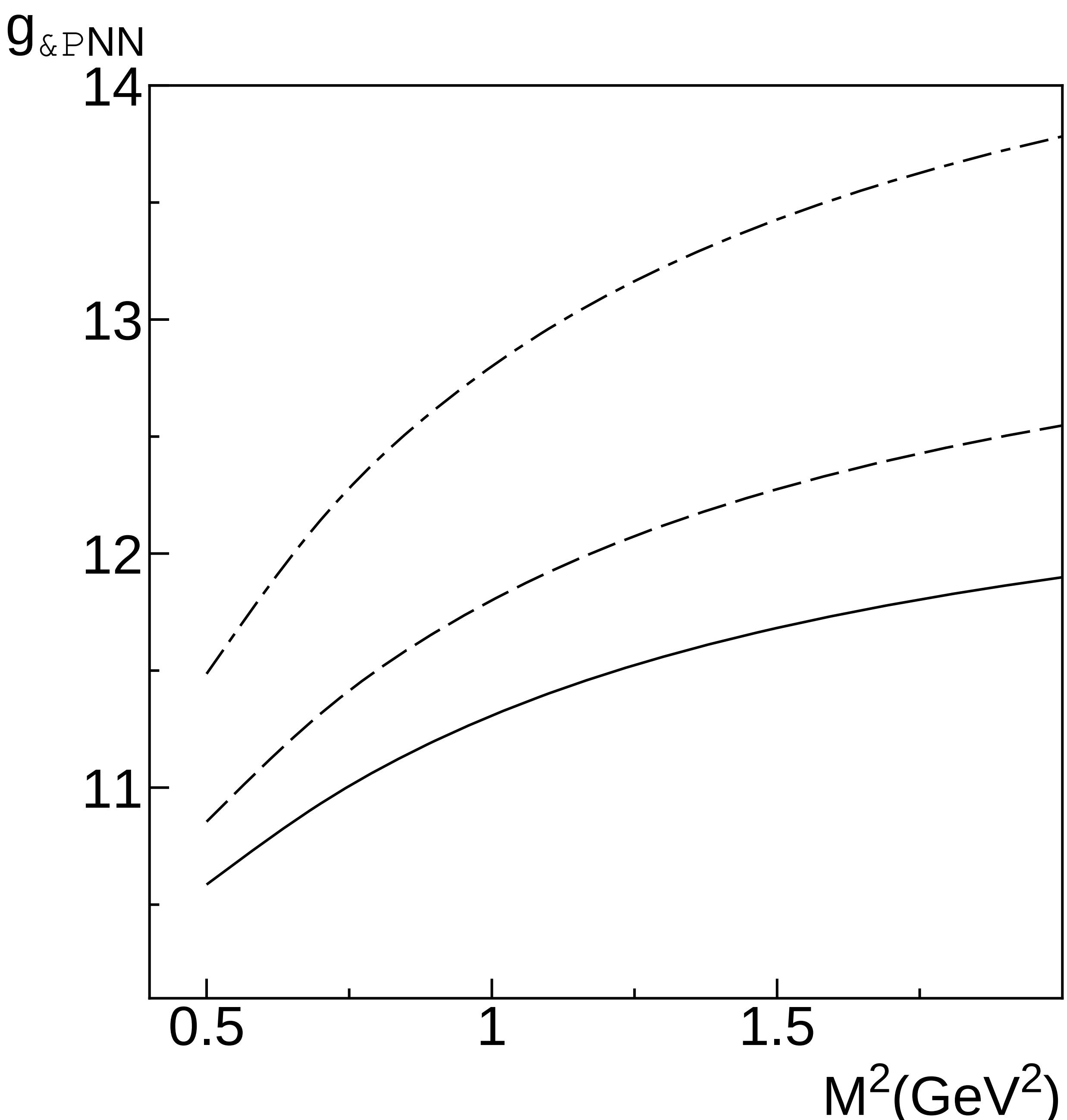


Fig.5(a)



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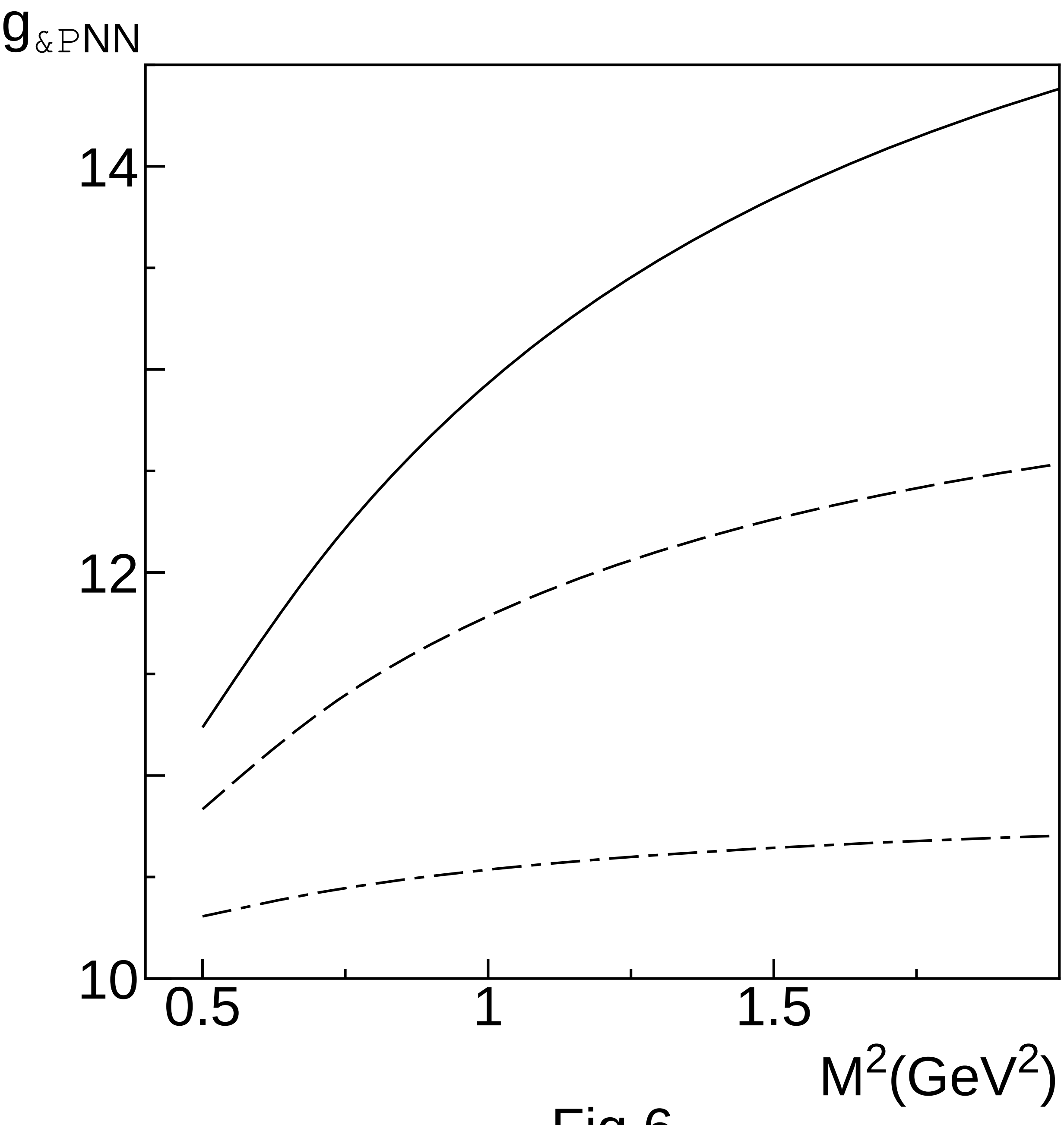

Fig.6

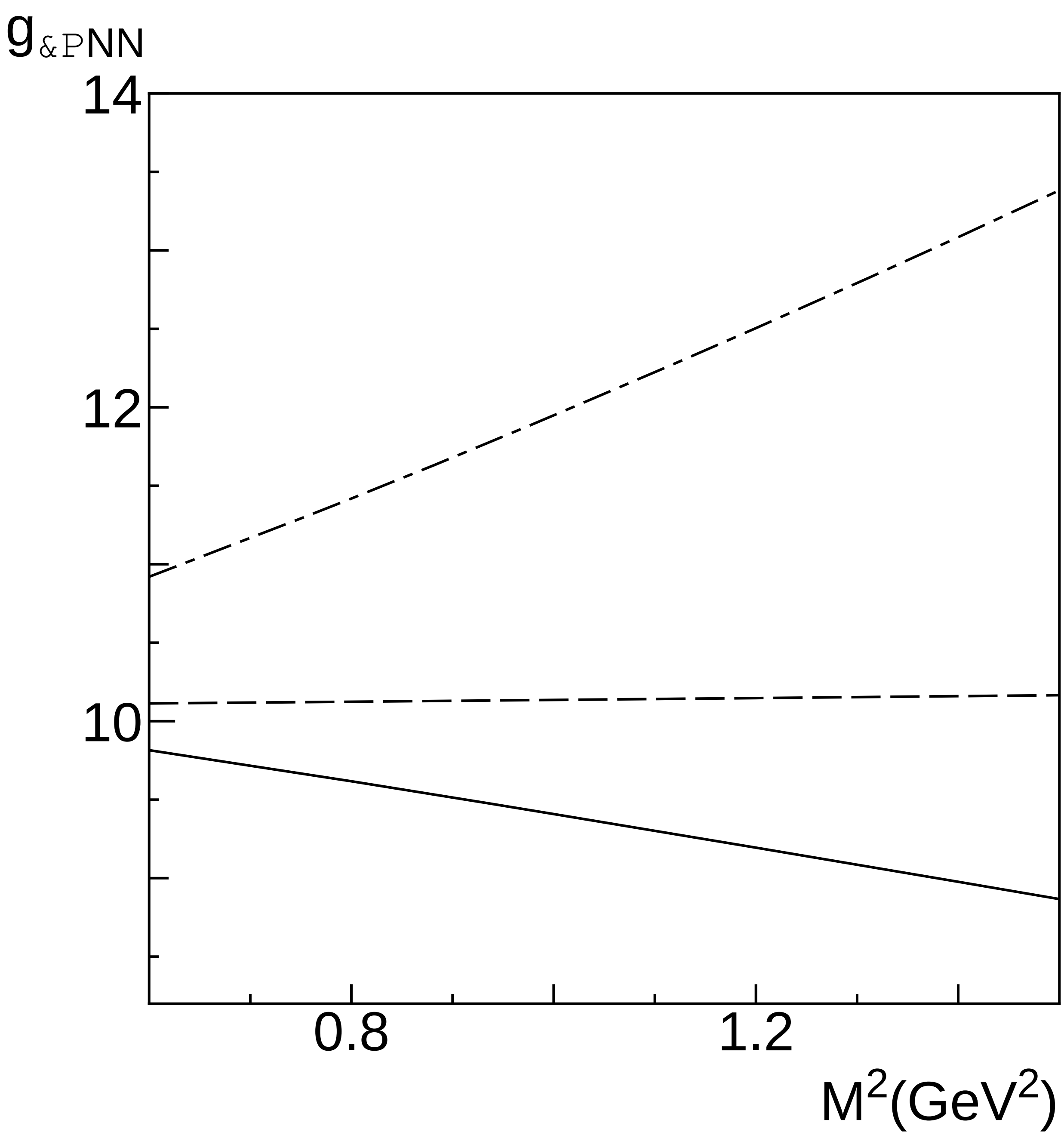

Fig.7